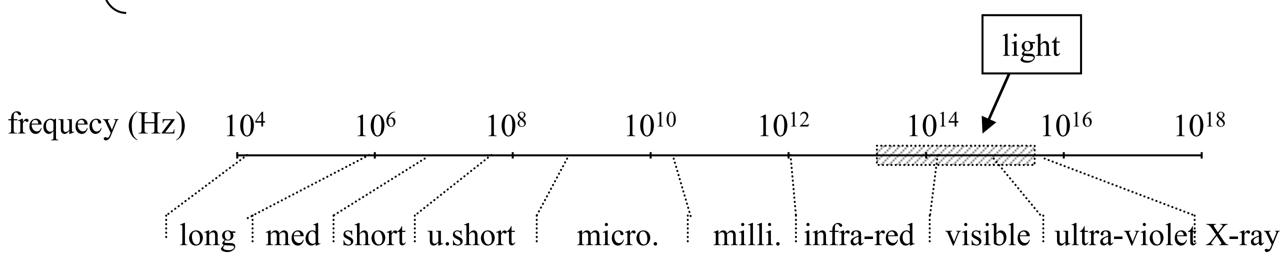
I. Photon

Light is an electromagnetic wave of a high frequency.

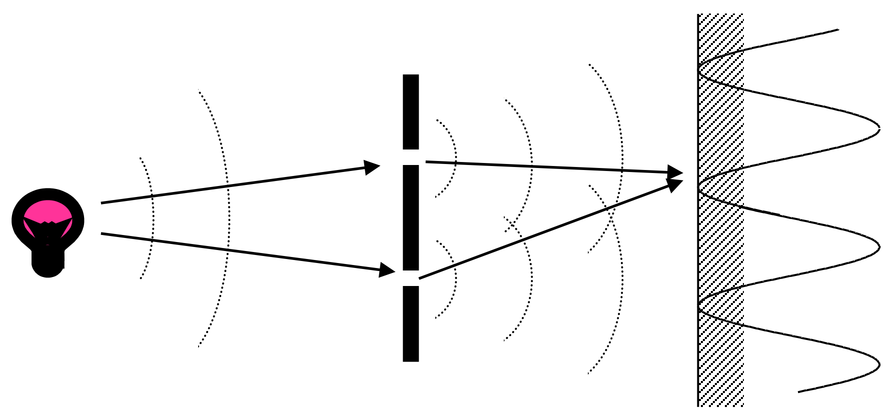
Maxwell's equation

$$\begin{cases} \nabla \times \mathbf{E} = -\mu_0 \dfrac{\partial \mathbf{H}}{\partial t} \\ \nabla \times \mathbf{H} = \varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \\ \nabla \cdot \mathbf{E} = 0 \\ \nabla \cdot \mathbf{H} = 0 \end{cases} \Rightarrow \nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} \Rightarrow E = A\cos(kz - \omega t + \theta)$$

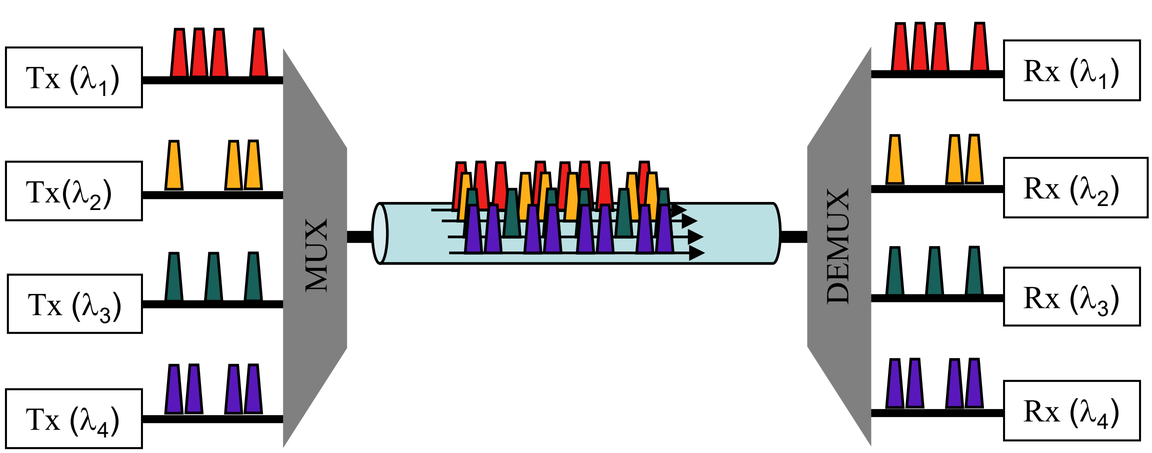
propagating electrical field while oscillating



There are various phenomena arising from wave properties, e.g., interference.



Photonic networks utilize wave properties of light.



However,,,,,

There are some physical phenomena that cannot be explained by the wave picture.

↓

Hypothesis: “There is a minimum unit of the light energy that cannot be divided furthermore.”

$\boxed{\text{photon}}$ (analogous to “electron”)

↓

The birth of Quantum Mechanics

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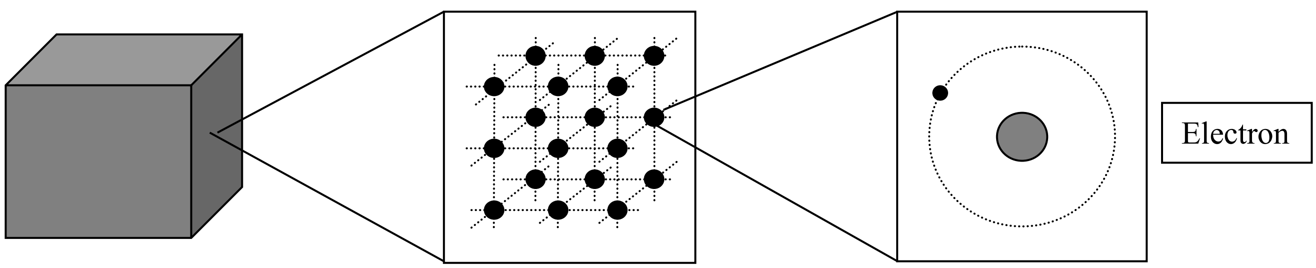
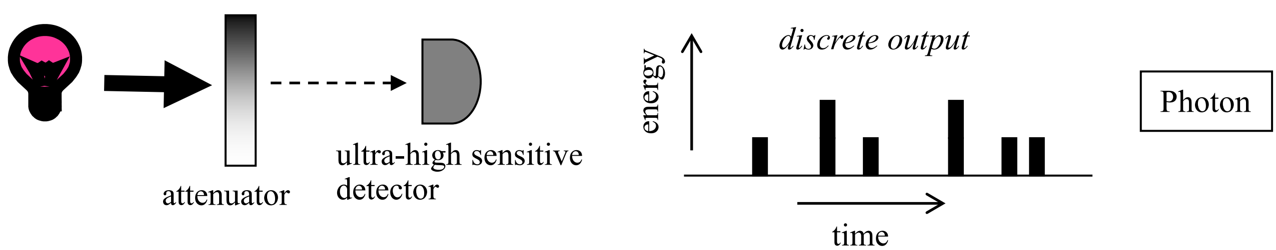
Advent of lasers, CCD cameras, etc.

<u>Matter</u>

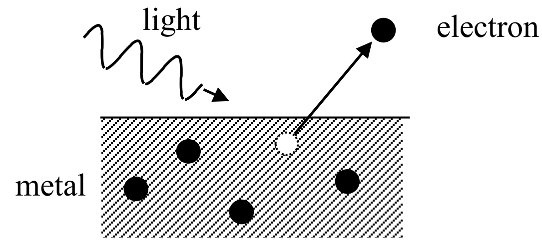


<u>Light</u>

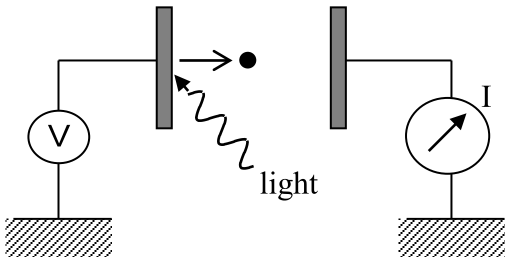


This chapter introduces phenomena that suggest ‘*photon*’.

Phenomenon of electrons jumping out from a metal surface eliminated by light

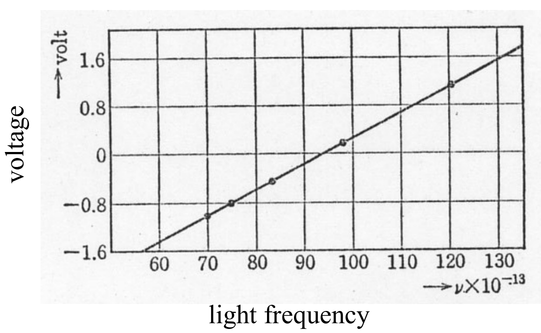


Millikan's experiment



- Voltage at which the current starts to flow is measured.
  The initial motion energy of an electron emitted from the metal can be evaluated
- The current is measured.
  The number of electrons emitted from the metal can be evaluated.

(results)



- The electron energy is independent of the light intensity, but it depends on the frequency as $E = hf - P$
  (f: light frequency, h, P: constant)
- The number of electrons is proportional to the light intensity

These results cannot be explained by the wave model.

Then, Einstein thought ,,,,

- Light is an assemble of energy particles.
- The energy of a light particle is proportional to the light frequency.

$\boxed{\text{light quantum}}$ → photon

This hypothesis can explain the experimental results.

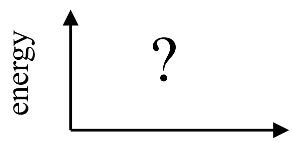
[Black body radiation]

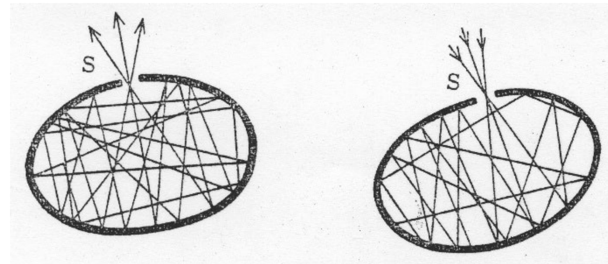
Heated matters radiate light, whose color (i.e., wavelength or frequency) depends on the temperature.

ex) molten iron

↓

How to theoretically explain its spectrum ?





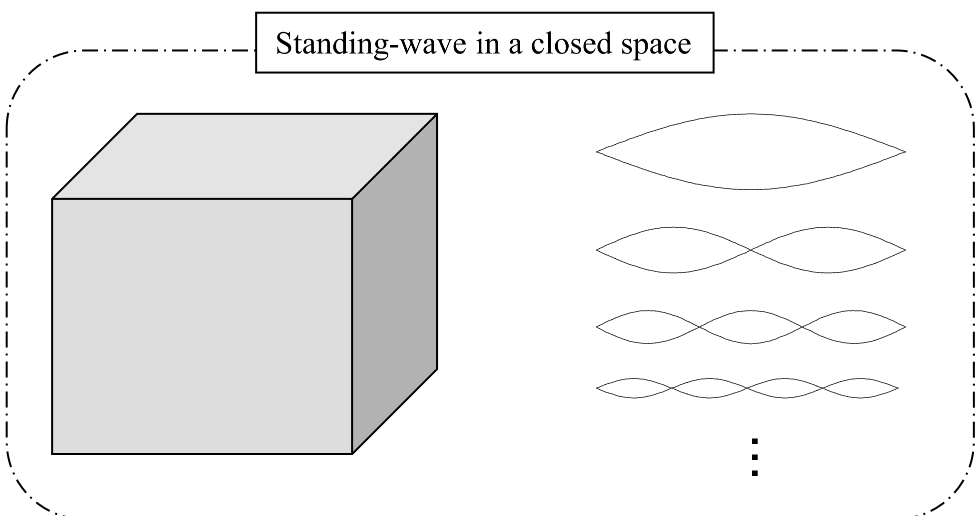
Black body radiation, or Cavity radiation
(radiation from a thermally equilibrium matter)

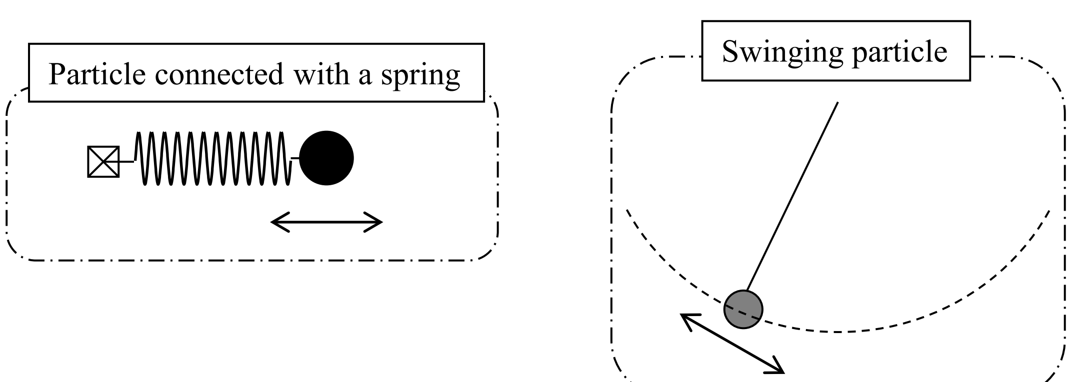
Rayleigh-Jeans' formula

A formula based on thermodynamics was proposed for explaining the energy spectrum of the radiation, which is introduced in this section.

The derivation starts with considering lightwave enclosed within a cavity.
The enclosed light forms a standing wave, which can be regarded as a harmonic oscillator.

(Harmonic oscillator ≡ a physical system that behaves sinusoidally)





The energy of an harmonic oscillator system is evaluated as

(total energy) = (the mean energy of harmonic oscillators) × (the number of harmonic oscillators),

which will be respectively discussed in the following.

The mean energy of one harmonic oscillator

Mathematically, the mean value of a stochastic variable y is given by

$$< y >= \int y(x)P(x)dx \qquad \left(\begin{array}{l} y(x)\text{: a variable dependent of } x \\ P(x)\text{: the probability density of } x \end{array} \right)$$

Let us derive the mean energy of one harmonic oscillator, based on this formula.

Here, we consider a spring system as an harmonic oscillator, whose energy is given by

(kinetic energy) + (potential energy).

The kinetic energy E_m can be expressed as

$E_m = (1/2)mv^2 = ap^2$ $\left(\begin{array}{l} m\text{: mass, } v\text{: velocity} \\ p\text{: momentum, } a\text{: proportional constant} \end{array} \right)$

Generally, the probability for a physical system to have an energy of E follows a Boltzmann distribution.

$$P(E) \propto \exp\left(-\frac{E}{kT}\right) = \exp\left(-\frac{ap^2}{kT}\right) \xrightarrow[\text{(normalized)}]{} P(p) = \frac{\exp\left(-ap^2/kT\right)}{\int \exp\left(-ap^2/kT\right)dp}$$

$$\left(\begin{array}{l} k\text{: Boltzmann constant} \\ T\text{: absolute temperature} \end{array} \right)$$

$$\Downarrow$$

$$< E_m >= \int ap^2 P(p)dp = \frac{\int ap^2 \exp\left(-ap^2/kT\right)dp}{\int \exp\left(-ap^2/kT\right)dp}$$

The integral in this equation is rewritten as

$$\int ap^2 \exp\left(-\frac{ap^2}{kT}\right)dp = -\frac{kT}{2}\int p\frac{d}{dp}\exp\left(-\frac{ap^2}{kT}\right)dp$$

$$= -\frac{kT}{2}\left[p\exp\left(-\frac{ap^2}{kT}\right)\right]_0^{\infty} + \frac{kT}{2}\int \exp\left(-\frac{ap^2}{kT}\right)dp = \frac{kT}{2}\int \exp\left(-\frac{ap^2}{kT}\right)dp$$

$$\Downarrow$$

$$< E_m >= \frac{(kT/2)\int \exp\left(-ap^2/kT\right)dp}{\int \exp\left(-ap^2/kT\right)dp} = \frac{kT}{2}$$

On the other hand, the potential energy of a particle connected to a spring is

$E_p = bq^2$ $\left(\begin{array}{l} q\text{: position} \\ b\text{: proportional constant} \end{array} \right)$ $\left(\int_0^q kx\,dx = \frac{kq^2}{2} \right)$

This expression is similar to that of the kinetic energy. Thus,

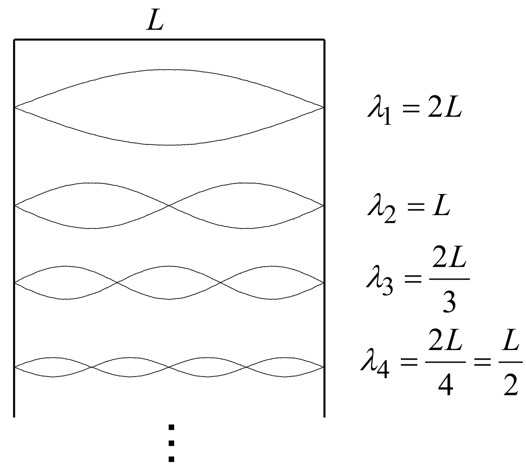
$$< E_{\mathrm{p}} >= \frac{kT}{2}$$

Subsequently, the average of the total energy of a particle fixed with a spring is expressed as

$$< E_{\mathrm{m}} > + < E_{\mathrm{p}} >= \frac{kT}{2} + \frac{kT}{2} = kT$$

<u>The number of harmonic oscillators</u>

Standing wave in a cavity (closed space) can be regarded as an harmonic oscillator.
Let us evaluate the number of standing waves in a cavity.



The wavelength of a one-dimensional standing wave is given by

$$\lambda = \frac{2L}{s} \qquad \left(\begin{array}{l} L\text{: cavity length} \\ s\text{: natural number} \end{array} \right)$$

In terms of frequency, $(\lambda f = c)$

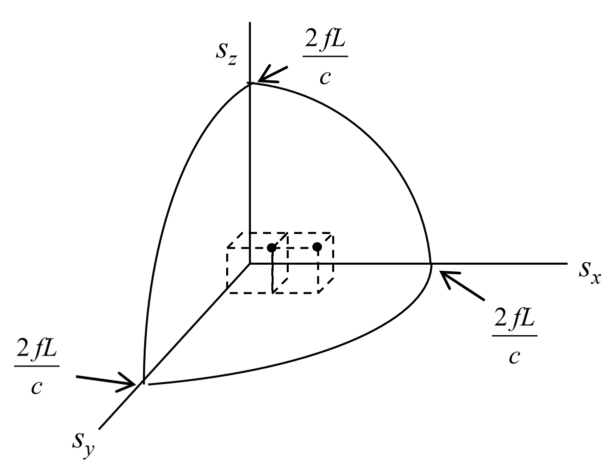
$$f = s\frac{c}{2L} \qquad \left(c\text{: light velocity} \right)$$

From this expression, the frequency of a standing wave in a cube (i.e., three dimension) is

$$f = \sqrt{f_x^2 + f_y^2 + f_z^2} = \sqrt{s_x^2 + s_y^2 + s_z^2}\,\frac{c}{2L} \quad \longrightarrow \quad \frac{2fL}{c} = \sqrt{s_x^2 + s_y^2 + s_z^2}$$

For a given L, the frequency of a standing wave is indicated by a set of natural numbers $\{s_x, s_y, s_z\}$.

The number of $\{s_x, s_y, s_z\}$ within a frequency ranges from f to $f + \delta f$ equals to the number of standing waves within this frequencies range.
The number of $\{s_x, s_y, s_z\}$ within frequency f = the number of lattice points on the surface of a (1/8)-sphere with a radius of $2fL/c$.
The number of standing waves within a frequency from 0 to f = the number of lattice points within a (1/8)-sphere with a radius of $2fL/c$.



In case that the lattice points are quite dense, we can approximate 'the number of lattice point = the volume.'

Therefore, the number of harmonic oscillators g is given by

(the degree of the freedom of the polarization state)

$$g = \frac{4}{3}\pi\left(\frac{2fL}{c}\right)^3 \times \frac{1}{8} \times 2 = \frac{8\pi f^3 L^3}{3c^3}$$

(density per a unit volume)

$$dg = \frac{8\pi f^2}{c^3} df \longrightarrow g_\nu = \frac{8\pi f^2}{c^3}$$

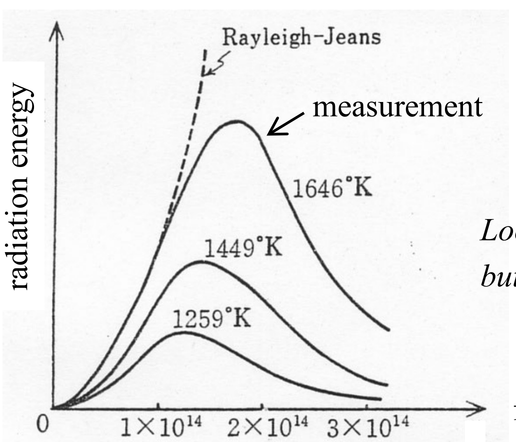
the number of harmonic oscillators per frequency

From the above considerations, the energy of harmonic oscillators per unit frequency (energy density) is

$$\text{(energy deinsity)} = \text{(average energy)} \times \text{(number per freq.)} = kT \times \frac{8\pi f^2}{c^3} = \frac{8\pi f^2 kT}{c^3}$$

Rayleigh-Jeans' formula

(diverging at high frequencies ?)



Looks good at low frequencies, but no good at high frequencies.

Planck's hypothesis

The unreasonable property of Rayleigh-Jeans' formula at high frequencies comes from '*the mean energy of harmonic oscillators = kT*', which is derived from

$$\langle E \rangle = \langle ap^2 + bq^2 \rangle \propto \underbrace{\iint (ap^2+bq^2)\exp\left(-\frac{ap^2+bq^2}{kT}\right) dq dp}$$

integral calculus assuming the energy as a continuous variable

reconsider

Planck thought "the energy is a discrete variable with a minimum unit ε".

$$E = n\varepsilon \qquad n = 0, 1, 2, 3, \cdots$$

energy quanta

Thus, $$< E >= \frac{\int E \exp(-E/kT)dE}{\int \exp(-E/kT)dE} \xrightarrow[\left(\begin{array}{c}\text{continuous} \to \text{discrete} \\ \int \to \sum\end{array}\right)]{} < E >= \frac{\sum_{n}^{\infty} n\varepsilon \exp(-n\varepsilon/kT)}{\sum_{n}^{\infty} \exp(-n\varepsilon/kT)}$$

$$\begin{cases} \text{numerator:} \quad \sum_{n}^{\infty} n\varepsilon e^{-n\varepsilon/kT} = -\dfrac{\partial}{\partial\left(\dfrac{1}{kT}\right)} \sum_{n}^{\infty} e^{-n\varepsilon/kT} = -\dfrac{\partial}{\partial\left(\dfrac{1}{kT}\right)} \dfrac{1}{1-e^{-\varepsilon/kT}} = \dfrac{\varepsilon e^{-\varepsilon/kT}}{(1-e^{-\varepsilon/kT})^2} \\ \text{denominator:} \quad \sum_{n}^{\infty} e^{-n\varepsilon/kT} = \dfrac{1}{1-e^{-\varepsilon/kT}} \end{cases}$$

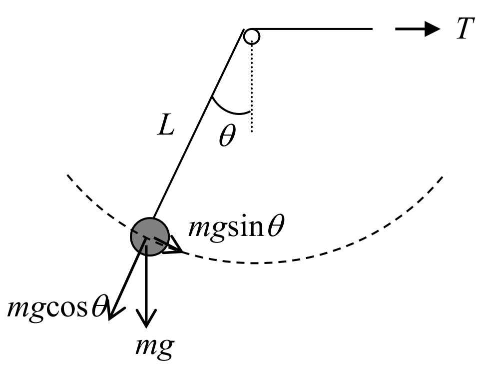
$$\Downarrow$$

$$< E >= \frac{\varepsilon e^{-\varepsilon/kT}}{1-e^{-\varepsilon/kT}} = \frac{\varepsilon}{e^{\varepsilon/kT}-1} \qquad \boxed{\text{Planck's law}}$$

About ε

In the above, we assume the minimum unit of light energy ε.
As a matter of fact, ε is proportional to the light frequency, which is suggested by the following consideration.

Here, we consider a swinging particle as an example of a harmonic oscillator.
Suppose that we slowly shorten the length of the string connected to a swinging particle.



Equation of motion: $mL\ddot{\theta} = -mg\sin\theta$

$$\downarrow$$

$$\theta = a\cos(2\pi ft + \delta) \qquad (a,\ \delta\text{: constant})$$

with $f \equiv \dfrac{1}{2\pi}\sqrt{\dfrac{g}{L}}$

$$\Downarrow$$

$$\frac{df}{dL} = -\frac{1}{2\pi L}\sqrt{\frac{g}{L}} \longrightarrow \delta f = -\frac{1}{2\pi L}\sqrt{\frac{g}{L}}\delta L = -\frac{\nu}{L}\delta L$$

Here, we consider the pulling force T that shortens the length L, which is expressed as

$$T = mg\cos\theta + mL\dot{\theta}^2 \approx mg(1-\frac{1}{2}\theta^2) + mL\dot{\theta}^2$$

(gravity along the length) (centrifugal force)

$$= mg - mga^2\{\frac{1}{2}\cos^2(2\pi ft+\varphi) - \sin^2(2\pi ft+\varphi)\}$$

The workload conducted by T (= the energy increment of the swinging particle) for the length to be shortened by δL is

$$<T>(-\delta L) = <mg - mga^2\{\frac{1}{2}\cos^2(2\pi ft+\varphi) - \sin^2(2\pi ft+\varphi)\}>(-\delta L) = -mg\delta L - \frac{mga^2}{4}\delta L$$

where, $< >$ denotes the temporal average.

(increment of the potential energy of the whole system)

(increment of the oscillation energy)

Thus, the change of the swinging energy is

$$\delta E = -\frac{mga^2}{4}\delta L$$

On the other hand, the energy of the swinging particle is

$$E = \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mgL(1-\cos\theta)$$

(kinetic energy) (potential energy from the lowest position)

$$\approx \frac{1}{2}mL^2\dot{\theta}^2 + \frac{1}{2}mgL\theta^2 = \frac{mga^2}{2}L$$

$$\left(\begin{array}{l}\theta = a\cos(2\pi ft+\varphi)\\ f = (1/2\pi)\sqrt{g/L}\end{array}\right)$$

$$\delta E = -\frac{E}{2L}\delta L \qquad \delta f = -\frac{f}{L}\delta L$$

$$\frac{\delta f}{\delta E} = \frac{f}{L}\frac{L}{2E}$$

$$\frac{\delta f}{f} = \frac{\delta E}{E}$$

$$\left(\begin{array}{l}d(E/f) = \delta E/f - (E/f^2)\delta\nu\\ \qquad = (E/f)(\delta E/E - \delta f/f) = 0\end{array}\right)$$

$$\boxed{\frac{E}{f} = const}$$

The bottom line is; "*energy is proportional to frequency.*"

$\boxed{\varepsilon = hf \quad (h\text{: proportional const.})}$ h: Planck constant

Planck's law, again

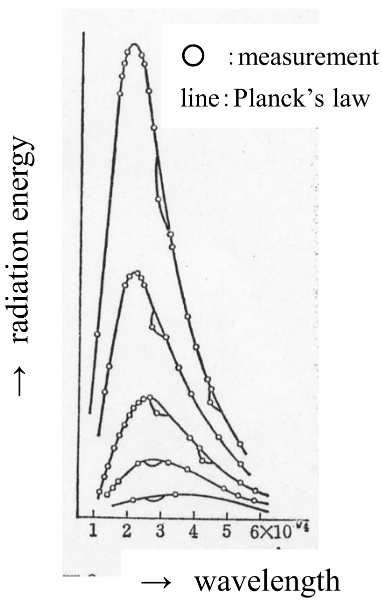
Substituting $\varepsilon = hf$ into the previous Planck's law

$$<E> = \frac{hf}{e^{hf/kT} - 1}$$

h is experimentally evaluated.
($h = 6.62 \times 10^{-27}$ erg-second)

Then, the energy density [= (average energy) × (density of harmonic oscillators)] is given by

Energy density: $$\frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/kT} - 1}$$



Good agreement between the theory and measurement

Why the classical model is OK at low frequencies ?

The minimum unit of energy is hf.

When $hf << kT$, the minimum unit is so small that it is regarded to be continuous.

Then, Rayleigh-Jeans' formula that treats the energy as continuous is OK at low-frequencies.

In case of $hf << kT$,

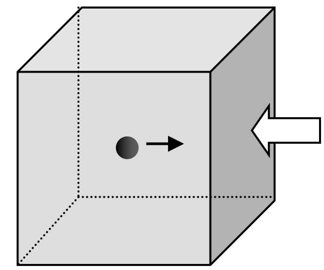
$$\frac{8\pi f^2}{c^3}\frac{hf}{e^{hf/kT} - 1} \approx \frac{8\pi f^2}{c^3}\frac{hf}{1 + hf/kT - 1} = \frac{8\pi f^2}{c^3}kT$$

Rayleigh-Jeans' Formula

Momentum of photon

Provided that light has a particle-like property, it can have ‘momentum’.

In this section, we discuss the momentum of a photon.



Our discussion here utilizing the cavity radiation in a cube.

Situation assumed is;

Suppose we slowly change the length of one side of a cube L.

Suppose one photon propagates in the direction of the changing side, which pushes the surface when bumping against it.

We move the surface against the pressure from the photon..

In this situation, the energy conservation rule tells us

(change of the photon energy) = (work of moving the surface by the external force)

= (pressure by the photon) × (moving length of the surface).

We will evaluate each term in the above equation.

(change of the photon energy)

The relationship between L and the frequency of a standing-wave is

$$f = s\frac{c}{2L} \longrightarrow \frac{df}{dL} = -s\frac{c}{2L^2} \longrightarrow \delta f = -s\frac{c}{2L}\frac{\delta L}{L} = -f\,\delta L$$

(s: natural number)

Photon energy is $E = hf$, thus

$$\frac{dE}{df} = h \longrightarrow \delta E = h\delta f \longrightarrow \delta E = -\frac{hf}{L}\delta L$$

(pressure by the photon)

Let us denote the momentum of a photon as p, then

- the change of the momentum when bumped back at the surface is $2p$

||

the pressure that the photon gives the surface at one bumping

- the frequency of the bumping is $c/2L$.

↓

The pressure that the photon gives to the surface is $2p \times \frac{c}{2L} = \frac{pc}{L}$

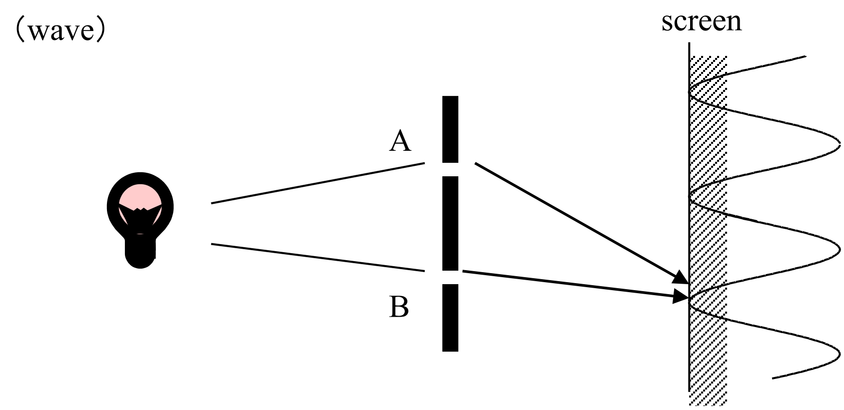
Therefore,

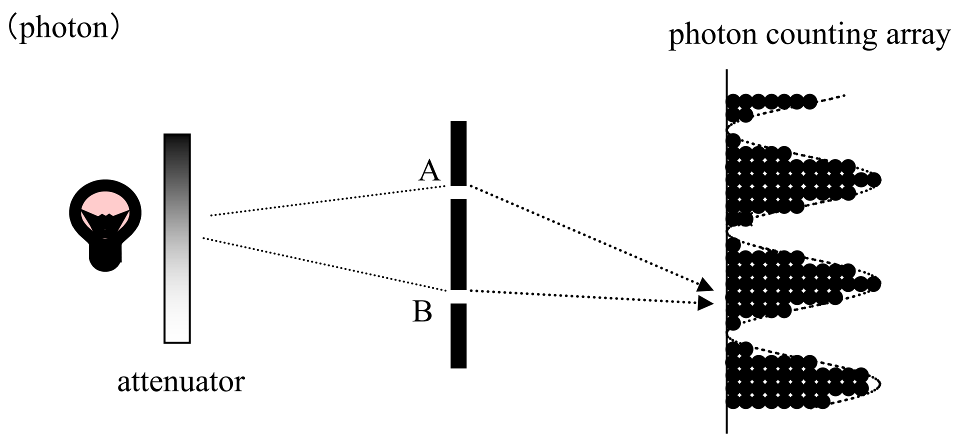
(change of the photon energy) = (pressure by the photon) × (moving length of the surface)

$$-\frac{hf}{L}\delta L = -\frac{pc}{L}\delta L \qquad \boxed{p = \frac{hf}{c}}$$ Momentum of a photon

[Wave property and particle-like property]

Light has particle-like properties, as described in this chapter.
However, there definitely exist phenomenon showing wave properties (e.g., interference).
How these two properties (particle and wave) are compatible ?
The answer is that the energy of light has a minimum unit whose behavior follows wave properties.





How the wave and the particle-like properties are compatible will be theoretically described in the next chapter. Brief summaries in advance are

- The state of light is expressed by the probability amplitude of photons: a
- The probability amplitude behaves like wave (i.e., it has a phase): $a = |a|e^{i\theta}$
- When we observe light, its energy has a minimum unit and thus discrete: hf
- The probability of observing a photon is given by the absolute square of the probability amplitude: $|a|^2$
- When some probability amplitudes are overlapped, there occurs interference:

$$|a + b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}[ab^*]$$

Note: "Particle-*like* property" does not mean that light particles fly over a space. It just means the light energy has a minimum unit and nothing else.