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Quantum mechanical treatment of traveling light in an absorptive medium of two-level systems

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ABSTRACT

Quantum mechanical treatment of a light wave that propagates through an absorptive medium is presented. Unlike a phenomenological beam-splitter model conventionally employed to describe a traveling light in a lossy medium, the time evolution of the field operator is derived using the Heisenberg equation with the Hamiltonian for a physical system, where the light wave interacts with an ensemble of two-level systems in a medium. Using the obtained time-evolved field operators, the mean values and variances of the light amplitude and the photon number are evaluated. The results are in agreement with those obtained in the beam-splitter model, giving a logical theoretical basis for the phenomenological beam-splitter model.

1. Introduction

Quantum fluctuations or quantum noises are of fundamental interest in quantum optics, including inherent fluctuations in coherent states and squeezed states, vacuum fluctuations, and quantum-limited noise figures in phase-insensitive or phase-sensitive optical amplifiers. It is well known that quantum properties of a light wave are affected by propagating loss in a medium. Conventionally, traveling light in a dielectric medium was quantum mechanically treated with a quantum mechanical version of the Maxwell's equations that includes a phenomenological noise field operator or an operator phenomenologically representing an absorption phenomenon [1–7]. A spatial differential equation of the light field operator (annihilation operator) was derived from the quantum mechanical Maxwell's equations, by which quantum properties of a traveling light was analyzed. A beam-splitter model was also suggested similar to the spatial differential equation [8–15], in which a noise field operator is assumed to be overlapped onto the attenuated light field operator via a beam splitter representing a loss phenomenon. This beam splitter model has been widely utilized in considering quantum properties of light wave propagating in a lossy medium because of its simplicity.

In a beam-splitter model, the in-out relationship of the light field operator through a loss medium is expressed as [8,15]

$$\hat{a}_{out} = \tilde{t}\hat{a}_{in} + \tilde{r}\hat{a}_{vac} \tag{1}$$

with $|\tilde{t}|^2 + |\tilde{r}|^2 = 1$, where $\hat{a}_{in}$ and $\hat{a}_{out}$ are the field operators at the input and output of the medium, respectively, $\hat{a}_{vac}$ is the vacuum field (or noise field) operator, and $\tilde{t}$ and $\tilde{r}$ are the amplitude transmittance and reflectance of a beam splitter, respectively. The first term in Eq. (1) represents the attenuation of the incident light, and the second term represents a noise field overlapped with the incident light caused by a reaction of some loss mechanism. While this beam-splitter model is convenient and useful, it was phenomenologically presented, not directly derived from the first principles of quantum mechanics. To the best of the author's knowledge, a logical derivation from the fundamentals of the quantum mechanical theory has not been reported.

Based on the above background, this paper presents a quantum mechanical description of a light wave passing through an absorptive medium. Interactions between light and an ensemble of two-level systems are assumed to cause attenuation of a traveling light, and the space evolution of the light wave state is derived using the Heisenberg equation with the Hamiltonian for such a physical system. The results are in agreement with those obtained in the beam-splitter model, thus providing a theoretical justification of the phenomenological beam-splitter model.

2. Theoretical treatment

2.1. Time evolution of the field operator

We consider an absorptive medium, in which traveling light is attenuated through interaction with an ensemble of two-level

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systems [16]. The Hamiltonian for such a physical system is expressed as [15]

$$\hat{H} = \hbar\omega\hat{a}^{\dagger}\hat{a} + \sum_j \hbar\omega_m^{(j)}\hat{\pi}_j^{\dagger}\hat{\pi}_j + i\sum_j \hbar(\alpha_j\hat{\pi}_j^{\dagger}\hat{a} - \alpha_j^*\hat{\pi}_j\hat{a}^{\dagger}). \tag{2}$$

The first, second, and third terms represent the Hamiltonians for the light field, an ensemble of two-level systems in the medium, and the interaction between the light and the two-level systems, respectively. Here, $\hat{a}$ and $\hat{a}^{\dagger}$ are the annihilation and creation operators of light, respectively; $\hat{\pi} = |2><1|$ and $\hat{\pi}^{\dagger} = |2><1|$ are the transition operators of a two-level system in the medium with $|2>$ and $|1>$ denoting the upper and lower energy levels, respectively; $\hbar$ is the reduced Planck's constant; ω is the lightwave angular frequency; $\hbar\omega_m$ is the energy difference between the two levels; α_j is a proportional constant; and the subscript j labels the two-level systems.

The time evolution of the field operator and the transition operators is governed by the Hamiltonian $\hat{H}$ of the composite system through the Heisenberg equation:

$$\frac{d\hat{a}}{dt} = \frac{1}{i\hbar}[\hat{a}, \hat{H}] = -i\omega\hat{a} - \sum_j \alpha_j\hat{\pi}_j, \tag{3a}$$

$$\frac{d\hat{\pi}_j}{dt} = \frac{1}{i\hbar}\left[\hat{\pi}_j, \hat{H}\right] = -i\omega_m^{(j)}\hat{\pi}_j + \alpha_j(\hat{\pi}_j\hat{\pi}_j^{\dagger} - \hat{\pi}_j^{\dagger}\hat{\pi}_j)\hat{a}. \tag{3b}$$

These equations can be simplified by rewriting the operators as $\hat{a} \to \hat{a}(t)e^{-i\omega t}$ and $\hat{\pi}_j \to \hat{\pi}_j(t)e^{-i\omega_j t}$:

$$\frac{d\hat{a}}{dt} = -\sum_j \alpha_j^*\hat{\pi}_j e^{-i(\omega_j-\omega)t}, \tag{4a}$$

$$\frac{d\hat{\pi}_j}{dt} = \alpha_j(\hat{\pi}_j\hat{\pi}_j^{\dagger} - \hat{\pi}_j^{\dagger}\hat{\pi}_j)\hat{a}e^{i(\omega_j-\omega)t} = \alpha_j\hat{\Pi}_j\hat{a}e^{i(\omega_j-\omega)t}, \tag{4b}$$

where $\hat{\Pi}_j \equiv \hat{\pi}_j\hat{\pi}_j^{\dagger} - \hat{\pi}_j^{\dagger}\hat{\pi}_j$ is a shorthand notation.

We solve Eqs. (4) by employing an iterative approximation. First, the first-order solutions are obtained by substituting the initial values $\{\hat{a}^{(0)}, \hat{\pi}_j^{(0)}\}$ into the right-hand side of Eqs. (4):

$$\frac{d\hat{a}}{dt} = -\sum_j \alpha_j^*\hat{\pi}_j^{(0)} e^{-i(\omega_j-\omega)t}, \tag{5a}$$

$$\frac{d\hat{\pi}_j}{dt} = \alpha_j\hat{\Pi}_j^{(0)}\hat{a}^{(0)}e^{i(\omega_j-\omega)t}. \tag{5b}$$

The solutions of these equations are

$$\hat{a}(t) = \hat{a}^{(0)} - i\sum_j \alpha_j^*\frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}, \tag{6a}$$

$$\hat{\pi}(t) = \hat{\pi}_j^{(0)} - i\alpha_j\frac{e^{i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{a}^{(0)}\hat{\Pi}_j^{(0)}. \tag{6b}$$

Next, we substitute these first-order solutions into the right-hand side of Eq. (4a):

$$\begin{aligned}\frac{d\hat{a}}{dt} &= -\sum_j \alpha_j^*\left[\hat{\pi}_j^{(0)} - i\alpha_j\frac{e^{i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{a}^{(0)}\hat{\Pi}_j^{(0)}\right]e^{-i(\omega_j-\omega)t} \\ &= -\sum_j \alpha_j^*\hat{\pi}_j^{(0)}e^{-i(\omega_j-\omega)t} \\ &\quad - i\hat{a}^{(0)}\sum_j\left|\alpha_j\right|^2\frac{1}{\omega_j-\omega}\left\{e^{-i(\omega_j-\omega)t} - 1\right\}\hat{\Pi}_j^{(0)}.\end{aligned} \tag{7}$$

The solution of this equation is given by

$$\begin{aligned}\hat{a}(t) &= \hat{a}^{(0)}\left[1 - \sum_j\left|\alpha_j\right|^2\frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j-\omega)t}{(\omega_j-\omega)^2}\hat{\Pi}_j^{(0)}\right] \\ &\quad - i\sum_j \alpha_j^*\frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}.\end{aligned} \tag{8}$$

We assume the interaction time is short and consider the above expression of $\hat{a}(t)$ as the solution of Eq. (4a).

2.2. Physical quantities

Physical quantities of a light wave are expressed in terms of expectation values of $\hat{a}(t)$ in an initial state of the composite system under consideration. The mean amplitude is given by $<a> = <\Psi_0|\hat{a}(t)|\Psi_0>$, where $|\Psi_0>$ is an initial state of the composite system of light and medium. Here, we are considering an absorptive medium, not an amplifying one, and thus, we assume that all two-level systems are initially in the lower energy states. Such an initial state can be expressed as

$$\left|\Psi_0\right> = \left|\Psi_r^{(0)}\right> \otimes \left|\Psi_m^{(0)}\right> \tag{9}$$

with

$$|\Psi_m^{(0)}> = \bigotimes_j |1>_j, \tag{10}$$

where $|\Psi_r{}^{(0)}>$ and $|\Psi_m{}^{(0)}>$ denote the initial states of the light and the medium, respectively. Applying this initial state to the time-evolved field operator $\hat{a}(t)$ given by Eq. (8), we find that the mean amplitude of the light wave at time t is

$$<a(t)> = <a(0)>\left\{1 - \sum_j |\alpha_j|^2\frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j-\omega)t}{(\omega_j-\omega)^2}\right\}, \tag{11}$$

where $<a(0)> \equiv <\Psi_r^{(0)}\left|\hat{a}^{(0)}\right|\Psi_r^{(0)}>$, and $<\Psi_m^{(0)}\left|\hat{\pi}_j^{(0)}\hat{\pi}_j^{(0)\dagger}\right|\Psi_m^{(0)}>=1$, $<\Psi_m^{(0)}\left|\hat{\pi}_j^{(0)\dagger}\hat{\pi}_j^{(0)}\right|\Psi_m^{(0)}>=0$, and $<\Psi_m^{(0)}\left|\hat{\pi}_j^{(0)}\right|\Psi_m^{(0)}>=0$ have been used to obtain the result.

The second term in Eq. (11) includes information of the energy states in the medium, which can be simplified as follows. First, we decompose the second term into the real and imaginary parts as

$$\begin{aligned}&\sum_j |\alpha_j|^2\frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j-\omega)t}{(\omega_j-\omega)^2} \\ &= 2\sum_j |\alpha_j|^2\frac{\sin^2[(\omega_j-\omega)t/2]}{(\omega_j-\omega)^2} + i\sum_j |\alpha_j|^2\frac{\sin[(\omega_j-\omega)t] - (\omega_j-\omega)t}{(\omega_j-\omega)^2}.\end{aligned} \tag{12}$$

Under the condition that the energy states are densely distributed in the frequency domain, the summation in the real part can be replaced by an integral, i.e.,

$$2\sum_j |\alpha_j|^2 \frac{\sin^2[(\omega_j-\omega)t/2]}{(\omega_j-\omega)^2} = 2\int_{-\infty}^{+\infty} |\alpha(\omega+\Omega)|^2 \frac{\sin^2[\Omega t/2]}{\Omega^2}\rho(\omega+\Omega)d\Omega, \tag{13}$$

where ρ is the spectrum density of the energy states. Moreover, under the assumption that the spectral distribution of the energy states is sufficiently broad, this expression can be further simplified as

$$2\int_{-\infty}^{+\infty} |\alpha(\omega+\Omega)|^2 \frac{\sin^2[\Omega t/2]}{\Omega^2}\rho(\omega+\Omega)d\Omega \approx 2|\alpha_0|^2\rho_0 \int_{-\infty}^{+\infty} \frac{\sin^2[\Omega t/2]}{\Omega^2}d\Omega = |\alpha_0|^2\rho_0\pi t, \tag{14}$$

where $\alpha(\omega+\Omega)$ and $\rho(\omega+\Omega)$ are taken to be constants, which are denoted by α_0 and ρ_0, respectively. Note that the above development of the equation follows a conventional approach employed for analyzing light-atom interaction in a laser medium [15]. On the other hand, under the assumption $(\omega_j-\omega)t\gg 1$, the imaginary part of Eq. (12) can be rewritten as

$$\sum_j |\alpha_j|^2 \frac{\sin[(\omega_j-\omega)t]-(\omega_j-\omega)t}{(\omega_j-\omega)^2} \approx -\sum_j |\alpha_j|^2 \frac{(\omega_j-\omega)t}{(\omega_j-\omega)^2} = -t\sum_j \frac{|\alpha_j|^2}{\omega_j-\omega}. \tag{15}$$

Then, substituting Eqs. (13)–(15) into Eq. (11), we have

$$<a(t)> = <a(0)>\left\{1-\left(\frac{\alpha_t}{2}-i\eta_t\right)t\right\} \approx <a(0)> e^{(-\alpha_t/2+i\eta_t)t}, \tag{16}$$

where $\alpha_t \equiv 2|\alpha_0|^2\rho_0\pi$, $\eta_t \equiv \sum_j |\alpha_j|^2/(\omega_j-\omega)$, and the approximation of $(\alpha_t/2 - i\eta_t)t\ll 1$ have been used. Eq. (16) is comprehensive and corresponds to a classical description of traveling light in an absorptive medium. Using the relationship of $z/t=v$, where z and v are the propagation distance and light velocity, respectively, the above time evolution of the mean amplitude can be translated into a space evolution, yielding

$$<a(z_0+\Delta z)> = <a(z_0)> e^{(-\alpha/2+i\eta)\Delta z}, \tag{17}$$

where $\alpha \equiv \alpha_t/v$, $\eta \equiv \eta_t/v$, and z_0 is a spatial position at which the interaction takes place. Note that the parameter α corresponds to an attenuation coefficient in a classical expression.

The time evolution of the mean photon number $<n(t)>$ is also derived from $\hat{a}(t)$ as follows:

$$<n(t)> = <\Psi_0|\hat{n}(t)|\Psi_0> = <\Psi_0|\hat{a}^\dagger(t)\hat{a}(t)|\Psi_0> = <\Psi_r^{(0)}|\hat{a}^{(0)\dagger}\hat{a}^{(0)}|\Psi_r^{(0)}><\Psi_m^{(0)}|\left\{1-\sum_j |\alpha_j|^2 \frac{1-e^{i(\omega_j-\omega)t}+i(\omega_j-\omega)t}{(\omega_j-\omega)^2}\hat{\pi}_j^{(0)}\hat{\pi}_j^{(0)\dagger}\right\} \times \left\{1-\sum_j |\alpha_j|^2 \frac{1-e^{-i(\omega_j-\omega)t}-i(\omega_j-\omega)t}{(\omega_j-\omega)^2}\hat{\pi}_j^{(0)}\hat{\pi}_j^{(0)\dagger}\right\}|\Psi_m^{(0)}> \approx <n(0)>\left[1-\sum_j |\alpha_j|^2\left\{\frac{1-e^{-i(\omega_j-\omega)t}-i(\omega_j-\omega)t}{(\omega_j-\omega)^2}+\frac{1-e^{i(\omega_j-\omega)t}+i(\omega_j-\omega)t}{(\omega_j-\omega)^2}\right\}\right] = <n(0)>(1-\alpha_t t) \approx <n(0)> e^{-\alpha_t t}, \tag{18}$$

where $\hat{n} = \hat{a}^\dagger\hat{a}$ is the photon number operator and higher-order terms are ignored after the first line. Then, the space evolution of the mean photon number translated from Eq. (18) is

$$<n(z_0+\Delta z)> = <n(z_0)> e^{-\alpha\Delta z}. \tag{19}$$

Fluctuations or quantum noises of physical quantities are represented by variances. First, we consider the variance of the photon number, which is given by $\sigma_n^2 = <n^2> - <n>^2$. Using Eqs. (8) and (9), the average of the photon number squared is calculated as

$$<n(t)^2> = <\Psi_0|\hat{n}(t)^2|\Psi_0> = <\Psi_0|\hat{a}^\dagger(t)\hat{a}(t)\hat{a}^\dagger(t)\hat{a}(t)|\Psi_0> = <n(0)^2>(1-2\alpha_t t) + <n(0)> 2\sum_j |\alpha|^2 \frac{1-\cos[\omega_j-\omega)t]}{(\omega_j-\omega)^2} = <n(0)^2> e^{-2\alpha_t t} + <n(0)>(1-e^{-\alpha_t t}), \tag{20}$$

where higher-order terms are neglected, Eqs. (13) and (14) are used, and the approximations of $1-\alpha_t t \approx \exp(-\alpha_t t)$ and $1-2\alpha_t t \approx \exp(-2\alpha_t t)$ are applied. Using Eqs. (18) and (20), the variance of the photon number after the interaction with the absorptive medium is

$$\sigma_n^2(t) = <n(t)^2> - <n(t)>^2 = \left\{<n(0)^2> - <n(0)>^2\right\} e^{-2\alpha_t t} + <n(0)>(1-e^{-\alpha_t t}) = \sigma_n^2(0)e^{-2\alpha_t t} + <n(0)>(1-e^{-\alpha_t t}). \tag{21}$$

From this equation, the space evolution of the photon number variance is

$$\sigma_n^2(z_0+\Delta z) = \sigma_n^2(z_0)e^{-2\alpha\Delta z} + <n(z_0)> e^{-\alpha\Delta z}(1-e^{-\alpha\Delta z}). \tag{22}$$

Next, we consider fluctuations in the light wave amplitude. The light amplitude is composed of two quadratures, i.e., the real and imaginary parts or the cosine and sine components, and the amplitude fluctuations are usually expressed in terms of fluctuations in the two quadratures. The operators representing the real and imaginary parts of the light wave amplitude are given by $(\hat{a}+\hat{a}^\dagger)/2$ and $(\hat{a}-\hat{a}^\dagger)/2i$, respectively. However, we modify these expressions as follows in order to simplify the derivation.

The time evolution of the mean amplitude shown in Eq. (16) indicates that the phase of the light amplitude rotates by $\eta_t t$ through the interaction with the medium. Because a global phase rotation does not affects physical properties, we can redefine the field operator to factor out this global phase rotation from the field operator. In other words, we define $\hat{b} \equiv \hat{a}\exp(-i\eta_t t)$ as a modified field operator for considering amplitude fluctuations. The real part of this field operator is

$$\hat{x}_1(t) = \frac{\hat{b}(t)+\hat{b}^\dagger(t)}{2} = \frac{1}{2}\{\hat{a}(t)e^{-i\eta_t t} + \hat{a}^\dagger(t)e^{i\eta_t t}\}, \tag{23}$$

and its average is

$$<x_1(t)> = \frac{1}{2}\{ < a(0) > + < a^*(0) > \}e^{-(\alpha_t/2)t} = < x_1(0) > e^{-(\alpha_t/2)t}, \tag{24}$$

where Eq. (16) has been used. On the other hand, the average of the square is

$$<x_1(t)^2 > = < x_1(0)^2 > e^{-\alpha_t t} + \frac{1}{4}(1 - e^{-\alpha_t t}), \tag{25}$$

where the approximations employed in the previous derivations have been applied. Then, the variance of the real part of the light amplitude is obtained from Eqs. (24) and (25) as

$$\sigma_{x1}^2(t) = < x_1(t)^2 > - < x_1(t) >^2 = < x_1(0)^2 > e^{-\alpha_t t} + \frac{1}{4}(1 - e^{-\alpha_t t}) - < x_1(0) >^2 e^{-\alpha_t t} = \sigma_{x1}^2(0)e^{-\alpha_t t} + \frac{1}{4}(1 - e^{-\alpha_t t}), \tag{26}$$

which can be translated to

$$\sigma_{x1}^2(z_0 + \Delta z) = \sigma_{x1}^2(z_0)e^{-\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z}). \tag{27}$$

The space evolution of the variance of the imaginary part $\hat{x}_2 = (\hat{b} - \hat{b}^\dagger)/2i$ is obtained similarly, yielding

$$\sigma_{x2}^2(z_0 + \Delta z) = \sigma_{x2}^2(z_0)e^{-\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z}). \tag{28}$$

2.3. Light propagation over a distance in a medium

In the previous section, we discuss time or space evolution during a short time. For light wave propagating over a distance in a medium, however, the assumptions in the previous section are not necessarily satisfied. In addition, the light wave successively interacts with new two-level systems while propagating, the effect of which is not taken into account in the previous section. Therefore, we discuss in this section the space evolution of a light wave propagating over a distance in an absorptive medium.

In order to analyze the above situation, we divide a light propagation distance in the medium into small segments, and assume that the time evolution of the field operator derived in the previous section, i.e., $\hat{a}(t)$ given by Eq. (8), is applicable in each segment. In each segment, the two-level systems are assumed to be initially at the lower energy states as expressed in Eq. (10). This assumption simulates the situation that the traveling light successively interacts with new two-level systems in each segment.

Under the above assumptions, using Eq. (17), the mean amplitude at the end of the kth segment is expressed as

$$<a(z_{k+1}) > = < a(z_k) > e^{(-\alpha/2+i\eta)\Delta z}, \tag{29}$$

where z_k is a spatial position at the input of the kth segment, and Δz is the length of one segment. Iteratively applying the above equation to each successive segment of the whole propagation distance in the medium, we obtain the mean amplitude at the output of the medium as

$$<a(z_{out}) > = < a(z_N) > e^{(-\alpha/2+i\eta)\Delta z} = < a(z_{N-1}) > e^{2(-\alpha/2+i\eta)\Delta z} = \cdots = < a(z_{in}) > e^{(-\alpha/2+i\eta)L}, \tag{30}$$

where z_{in} and z_{out} are the position of the input and output of the medium, respectively, N is the number of segments, and $L \equiv N \times \Delta z$ is the propagation distance. Similarly, the mean photon number at the output of the medium is obtained by using Eq. (19), yielding

$$<n(z_{out}) > = < n(z_N) > e^{-\alpha\Delta z} = \cdots = < n(z_{in}) > e^{-\alpha L}. \tag{31}$$

Eqs. (30) and (31) are in agreement with the classical expressions for the amplitude and intensity of traveling light, respectively.

Similarly, the variance of the photon number at the output of the medium is obtained by using Eqs. (22) and (31):

$$\begin{aligned}\sigma_n^2(z_{out}) &= \sigma_n^2(z_N)e^{-2\alpha\Delta z} + < n(z_N) > e^{-\alpha\Delta z}(1 - e^{-\alpha\Delta z}) = \left\{ \sigma_n^2(z_{N-1})e^{-2\alpha\Delta z} + < \hat{n}(z_{N-1}) > e^{-\alpha\Delta z}(1 - e^{-\alpha\Delta z})\right\}e^{-2\alpha\Delta z} + < \hat{n}(z_{N-1}) > e^{-2\alpha\Delta z}(1 - e^{-\alpha\Delta z}) \\ &= \sigma_n^2(z_{N-1})e^{-4\alpha\Delta z} + < \hat{n}(z_{N-1}) > e^{-2\alpha\Delta z}(1 - e^{-2\alpha\Delta z}) \\ &\vdots \\ &= \sigma_n^2(z_{in})e^{-2\alpha L} + < n(z_{in}) > e^{-\alpha L}(1 - e^{-\alpha L}).\end{aligned} \tag{32}$$

Moreover, the variance of the quadrature of the amplitude is found by using Eq. (27) and the result is

$$\begin{aligned}\sigma_{x1}^2(z_{out}) &= \sigma_{x1}^2(z_N)e^{-\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z}) \\ &= \left\{ \sigma_{x1}^2(z_{N-1})e^{-\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z})\right\}e^{-\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z}) \\ &= \sigma_{x1}^2(z_{N-1})e^{-2\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z})(1 + e^{-\alpha\Delta z}) = \sigma_{x1}^2(z_{N-n}) e^{-(n+1)\alpha\Delta z} + \frac{1}{4}(1 - e^{-\alpha\Delta z})\sum_{k=0}^{n} e^{-k\alpha\Delta z} = \sigma_{x1}^2(z_{in}) e^{-\alpha L} + \frac{1}{4}(1 - e^{-\alpha L}).\end{aligned} \tag{33}$$

The expression for the variance of the quadrature ${\sigma_{x2}}^2$ is similar to this expression.

Eqs. (32) and (33) indicate well-known quantum noise properties of a light wave suffering from propagation loss, as follows. When the incident light is in a coherent state, for example, the initial variance of the photon number equals to the mean photon number, i.e., ${\sigma_n}^2(z_{in}) = < n(z_{in}) >$. Then, from Eqs. (32) and (31), the output variance is

$$\sigma_n^2(z_{out}) = < n(z_{in}) > e^{-2\alpha L} + < n(z_{in}) > e^{-\alpha L}(1 - e^{-\alpha L}) = < n(z_{out}) > . \tag{34}$$

The variance of photon number still equals to the mean photon number at the medium output. As for the light wave amplitude, the input variance of the quadrature $\hat{x}_1$ in a coherent state is ${\sigma_{x1}}^2(z_{in}) = 1/4$, and, from Eq. (33), its output variance is

$$\sigma_{x1}^2(z_{out}) = \frac{1}{4}e^{-\alpha L} + \frac{1}{4}(1 - e^{-\alpha L}) = \frac{1}{4}. \tag{35}$$

This equation shows that the variance of the quadrature $\hat{x}_1$ in a coherent state is constantly 1/4, regardless of propagation loss.

When a quadrature-squeezed state is incident, for another example, the input variances of the two quadratures are ${\sigma_{x1}}^2(z_{in}) = e^{-2s}/4$ and ${\sigma_{x2}}^2(z_{in}) = e^{2s}/4$, where s is the squeeze parameter [15]. From Eq. (33), the output variances are

$$\sigma_{x1}^2(z_{out}) = \frac{1}{4}e^{-2s}\cdot e^{-\alpha L} + \frac{1}{4}(1 - e^{-\alpha L}), \tag{36a}$$

$$\sigma_{x2}^2(z_{out}) = \frac{1}{4}e^{2s}\cdot e^{-\alpha L} + \frac{1}{4}(1 - e^{-\alpha L}). \tag{36b}$$

These expressions indicates that the variances changes from $e^{\pm 2s}/4$ to 1/4 as the quadrature-squeezed light propagates through an absorptive medium, suggesting that the squeezed state collapses to a coherent state as a result of medium loss. This is also a well-known quantum noise property of a light wave.

3. Discussion

In this section, we discuss how the present study is related to the conventional beam-splitter model [8–15]. In the beam-splitter model, the time evolution of the field operator is given by Eq. (1). The vacuum field operator $\hat{a}_{vac}$ appearing in that equation satisfies $<\hat{a}_{vac}> = <\hat{a}_{vac}^{\dagger}> = <\hat{a}_{vac}^{\dagger}\hat{a}_{vac}> = 0$ and $\left[\hat{a}_{vac}, \hat{a}_{vac}^{\dagger}\right] = 1$, and thus we have

$$<\tilde{r}\hat{a}_{vac}> = 0, \tag{37a}$$

$$<(\tilde{r}\hat{a}_{vac})^{\dagger}(\tilde{r}\hat{a}_{vac})> = 0, \tag{37b}$$

and

$$<(\tilde{r}\hat{a}_{vac})(\tilde{r}\hat{a}_{vac})^{\dagger}> = \left|\tilde{r}\right|^2 = 1 - \left|\tilde{t}\right|^2. \tag{37c}$$

On the other hand, time evolution of the field operator in the present treatment is given by Eq. (8). Let us consider the second term in Eq. (8), i.e.,

$$-i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}.$$

For an initial state given by Eq. (10), the averages of operators related to this term are

$$<\Psi_m^{(0)}\left|-i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}\right|\Psi_m^{(0)}> = 0, \tag{38a}$$

$$<\Psi_m^{(0)}\left|\left\{-i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}\right\}^{\dagger}\left\{-i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}\right\}\right| \Psi_m^{(0)}> = 0, \tag{38b}$$

and

$$<\Psi_m^{(0)}\left|\left\{-i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}\right\}\left\{-i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}\right\}^{\dagger}\right| \Psi_m^{(0)}> = \alpha_t t \approx 1 - e^{-\alpha_t t}, \tag{38c}$$

where Eqs. (13) and (14) and $\alpha_t = 2|\alpha_0|\rho_0\pi$ have been used to derive Eq. (38c). In addition, the average of the first term in Eq. (8) is

$$<\Psi_0\left|\hat{a}^{(0)}\left[1 - \sum_j |\alpha_j|^2 \frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j - \omega)t}{(\omega_j - \omega)^2}\hat{\Pi}_j^{(0)}\right]\right|\Psi_0> \approx <a(0)> e^{(-\alpha_t/2 + i\eta_t)t}, \tag{39}$$

where the calculation procedure for deriving Eq. (16) from Eq. (11) have been used.

The above expressions suggest that the transmitted signal term and the reflected vacuum term in Eq. (1) correspond to the first and second terms in Eq. (8), respectively, and we have

$$\tilde{t}\hat{a}_{in} \leftrightarrow \hat{a}^{(0)}\left[1 - \sum_j \left|\alpha_j\right|^2 \frac{1 - e^{-i(\omega_j-\omega)t} - i(\omega_j - \omega)t}{(\omega_j - \omega)^2}\hat{\Pi}_j^{(0)}\right], \tag{40a}$$

$$\tilde{r}\hat{a}_{vac} \leftrightarrow -i\sum_j \alpha_j^* \frac{e^{-i(\omega_j-\omega)t} - 1}{\omega_j - \omega}\hat{\pi}_j^{(0)}, \tag{40b}$$

with $|\tilde{t}|^2 = \exp(-\alpha_t t)$. In other words, the present theoretical treatment, which is based on the Heisenberg equation for a light-atom interacting system, provides a logical theoretical basis for the phenomenological beam-splitter model with the above correspondences.

4. Summary

This paper presented a quantum mechanical treatment of traveling light that propagates through an absorptive medium. Time evolution of the field operator was derived using the Heisenberg equation with the Hamiltonian for a physical system in which a light wave interacts with an ensemble of two-level systems. The mean values and variances of the light amplitude and the photon number were evaluated using the time-evolved field operator in an initial state of the system. The propagation effect of a light wave successively interacting with the two-level systems along its path was also taken into account. The derived results are in agreement with the conventionally employed phenomenological beam-splitter model, providing a justification of the beam-splitter model.

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